FYI - chain rule in several variables

$$\begin{tikzcd} \mathbb{R}^2_{(x,y)} \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}^3_{(a,b,c)} \arrow[r, "g"] & \mathbb{R}^2 \end{tikzcd}$$

Derivative matrix of

$f: \mathbb{R}^2 \to \mathbb{R}^3$

f' 3x2 matrix

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \\ (f_3)_x & (f_3)_y \end{pmatrix}$$

$g: \mathbb{R}^3 \to \mathbb{R}^2$

g': 2x3 matrix

$$g = \begin{pmatrix} g_1(a,b,c) \\ g_2(a,b,c) \end{pmatrix} \quad \begin{pmatrix} \text{Derivative} \\ \text{matrix} \\ \text{of } g \end{pmatrix} = g'(a,b,c)$$

$$g'(a,b,c) = \begin{pmatrix} (g_1)_a & (g_1)_b & (g_1)_c \\ (g_2)_a & (g_2)_b & (g_2)_c \end{pmatrix}$$

Chain Rule for $(g\circ f)'$:

$$(g\circ f)(x,y) = \begin{pmatrix} (g\circ f)_1(x,y) \\ (g\circ f)_2(x,y) \end{pmatrix}$$

$g\circ f: \mathbb{R}^2 \to \mathbb{R}^2$

$(g\circ f)'$ 2x2 matrix

$$(g\circ f)'_{(x,y)} = \underbrace{g'(f(x,y))}_{2\times 3} \cdot \underbrace{f'(x,y)}_{3\times 2} \leftarrow 2\times 2.$$

$$(g\circ f)'_{(x,y)} = \begin{pmatrix} (g_1)_a & (g_1)_b & (g_1)_c \\ (g_2)_a & (g_2)_b & (g_2)_c \end{pmatrix}_{\underset{a,\,b,\,c}{(f_1,f_2,f_3)(x,y)}} \begin{pmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \\ (f_3)_x & (f_3)_y \end{pmatrix}_{(x,y)}$$

$$= \begin{pmatrix} \frac{\partial g_1}{\partial a}\frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial b}\frac{\partial f_2}{\partial x} + \frac{\partial g_1}{\partial c}\frac{\partial f_3}{\partial x} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \end{pmatrix}.$$

$\frac{\partial g}{\partial x}$

This $\frac{\partial (g\circ f)_1}{\partial x}$

$\frac{\partial (g\circ f)_2}{\partial x}$

$\frac{\partial (g\circ f)_1}{\partial y}$

$\frac{\partial (g\circ f)_2}{\partial y}$

Derivative means $F : \mathbb{R}^k \to \mathbb{R}^n$ twice differentiable.

$$F(x) = F(a) + F'(a)(x-a) + \mathcal{O}(|x-a|^2)$$

Tangent plane approx

($\mathcal{O}(|x-a|^2)$ means a quantity Bubba such that $|\text{Bubba}| \le C|x-a|^2$ $\forall$ values of variable) where C is a constant.)

(o(blah) means a quantity Bubba s.t. $|\text{Bubba}| \le C|\text{blah}|$

mean $\dfrac{|\text{Bubba}|}{|\text{blah}|} \to 0$:)

$x \in \mathbb{R}^k$

(n vector) (n×k) k-dim vector

$$\Rightarrow \underset{\mathbb{R}^n}{\underset{\cap}{F(x)}} \approx F(a) + F'(a)\,\overbrace{(x-a)}$$

near $x = a$.

Let's apply this to Complex Analysis

Where $f : \overset{(z)}{\mathbb{C}} \to \mathbb{C}$

(or a domain $U \subseteq \mathbb{C}$)

"domain" ← open connected subset of $\mathbb{C}$

Let $\tilde{f} : \overset{(x,y)}{\mathbb{R}^2} \to \mathbb{R}^2$ be the same fcn as f but using the real coordinates.

$\tilde{f}$ differentiable at (x_0, y_0)

$$\Longleftrightarrow \tilde{f}(x,y) \approx \underset{2\times 1}{\tilde{f}(x_0,y_0)} + \underset{2\times 2}{\tilde{f}'(x_0,y_0)} \underset{2\times 1}{\begin{pmatrix} x - x_0 \\ y - x_0 \end{pmatrix}}$$

$$\tilde{f} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \approx \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(x_0,y_0)} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$\tilde{f}(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is real differentiable ~

1st row

$$\star \; u(x,y) \approx u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)$$

2nd row

$$\star\star \; v(x,y) \approx v(x_0,y_0) + v_x(x_0,y_0)(x-x_0) + v_y(x_0,y_0)(y-y_0)$$

$\mathbb{C}$ - differentiability for f

$$f(z) \approx f(a) + f'(a)(z-a),$$

where $a = x_0 + iy_0$, $z = x+iy$

$f(z) = (u(x,y) + i\, v(x,y))$

what does this say? about u & v?

$$u(x,y) + iv(x,y) \approx u(x_0,y_0) + i\, v(x_0,y_0) + \left(u_x(x_0,y_0) + i v_x(x_0,y_0)\right) \cdot \left((x-x_0) + i(y-y_0)\right)$$

$$f'(z) = u_x + i\, v_x$$

$$u(x,y) + i\,v(x,y) \approx u(x_0,y_0) + i\,v(x_0,y_0) + u_x(x_0,y_0)(x-x_0)$$
$$- v_x(x_0,y_0)(y-y_0)$$
$$+ \big(v_x(x_0,y_0)(x-x_0)\big)i + \big(u_x(x_0,y_0)(y-y_0)\big)i$$

Real part of both sides.

$$u(x,y) \approx u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) - v_x(x_0,y_0)(y-y_0)$$

Same as $\star$ if $-v_x = u_y$ at (x_0,y_0)

Imag part of both sides.

$$v(x,y) \approx v(x_0,y_0) + v_x(x_0,y_0)(x-x_0) + u_x(x_0,y_0)(y-y_0)$$

Same as $\star\star$ if $u_x = v_y$.

$\therefore$ in complex variables, the chain rule will work.

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{C}$$

$$(g \circ f)'(z) = g'(f(z))\,f'(z)$$

if f & g are $\mathbb{C}$-diff'ble at z, $f(z)$.

New notation:

$$z = x + iy$$

$$dz = dx + i\,dy$$ differential 1-forms

eat vectors in x & y. $(1,0)$, $(0,1)$

$\partial_x = \frac{\partial}{\partial x}$ $\partial_y = \frac{\partial}{\partial y}$

$$dx(\partial_x) = 1$$
$$dx(\partial_y) = 0$$
$$dy(\partial_x) = 0$$
$$dy(\partial_y) = 1$$

$$dx(3\partial_x + 2\partial_y) = 3$$
$$dy(3\partial_x + 2\partial_y) = 2$$

$$3\partial_x + 2\partial_y = (3,2).$$

$$dz = dx + i\,dy$$
$$d\bar{z} = dx - i\,dy$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Amazing Facts

1. $f'(z) = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv)$

(check — works by C-R eqns).

(2) $\frac{\partial f}{\partial \bar{z}} = 0 \iff$ Cauchy-Riemann eqns are satisfied

(3) $\frac{\partial (z^n)}{\partial z} = n z^{n-1}$

$\frac{\partial}{\partial \bar{z}}(\bar{z}^n) = n \bar{z}^{n-1}$ etc.

One more consequence of the C-R equations:

(Assume u, v are C^2)

$$u_x = v_y \Rightarrow u_{xx} = v_{yx}$$
$$u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$$

If v is C^2, then $v_{xy} = v_{yx}$.

$$\Rightarrow u_{xx} + u_{yy} = 0$$

$\Delta u = 0$ } i.e. u is harmonic.

Laplacian.

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Similarly, $v_{xx} + v_{yy} = 0$. $\underbrace{\Delta v = 0}_{\text{i.e. } v \text{ is harmonic.}}$.